

# Ermakov-Lewis Invariants and Reid Systems

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Reid's  $m$ th-order generalized Ermakov systems of nonlinear coupling constant  $\alpha$  are equivalent to an integrable Emden-Fowler equation. We obtain a general formula for the Ermakov-Lewis invariant from this perspective and use the Bessel solutions of a parabolic frequency parametric oscillator for an illustrative calculation in this case ( $m = 2$ ). Next, for higher-order Reid systems ( $m \geq 3$ ) we obtain a closed formula for the invariant and also discuss the parametric solutions of these systems through the integration of the Emden-Fowler equation.

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In 1971, Reid introduced in a short paper the following higher-order generalization of Ermakov systems [1]

$$\begin{cases} u_{\zeta\zeta} + \omega^2(\zeta)u = 0 \\ v_{\zeta\zeta} + \omega^2(\zeta)v = \alpha(u_1u_2)^{m-2}v^{1-2m}, \quad \alpha \neq 0, \quad m \neq 0, 1, \quad m \in \mathbb{N}, \end{cases} \quad (1)$$

where  $u_1$  and  $u_2$  are particular linear independent solutions of the homogeneous linear equation of system (1). The standard Ermakov systems are included as the particular case  $m = 2$ .

Reid has shown that the following nonlinear superposition

$$v(\zeta) = \left( u_1^m + \frac{\alpha}{(m-1)W^2} u_2^m \right)^{\frac{1}{m}} \quad (2)$$

is a solution to the nonlinear equation in (1), where  $W$  is the nonzero constant Wronskian given by

$$W[u_1(\zeta), u_2(\zeta)] = u_1u_{2\zeta} - u_2u_{1\zeta}. \quad (3)$$

In the particular case  $m = 2$ , Reid's formula (2) reduces to Pinney's formula [2] corresponding to Ermakov's systems.

As well known, for the standard Ermakov systems the Ermakov-Lewis (EL) quadratic invariant [3] is a very useful conserved quantity that in the last few years in the quantum-mechanical context, where it is known under the name of Lewis-Riesenfeld invariant [4], has been applied in such important areas as quantum control [5] and quantum computing [6].

Here, our main goal is to get general formulas both for the standard EL invariant and the higher order (Reid) invariants. For that, we apply Ermakov's idea [7] of eliminating  $\omega^2(\zeta)$  from (1), which leads to the equation:

$$\frac{d}{d\zeta}(uv_{\zeta} - vu_{\zeta}) = \alpha(u_1u_2)^{m-2}v^{1-2m}u. \quad (4)$$

Multiplying both sides by  $uv_{\zeta} - vu_{\zeta} = -v^2 \frac{d}{d\zeta} \left( \frac{u}{v} \right)$  we get

$$\frac{d}{d\zeta}(uv_{\zeta} - vu_{\zeta})^2 = -\alpha(u_1u_2)^{m-2}v^{4-2m} \frac{d}{d\zeta} \left( \frac{u}{v} \right)^2 \quad (5)$$

Since  $u_1$  and  $u_2$  are particular solutions that satisfy (3), then let us use

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$$u_2 = Wu_1 \int \frac{1}{u_1^2} d\zeta \quad (6)$$

which is the reduction of order formula. Substituting this into (5) it yields

$$\frac{d}{d\zeta}(uv_\zeta - vu_\zeta)^2 = -\frac{A}{m-1} \left( \int \frac{d\zeta}{u^2} \right)^{m-2} \frac{d}{d\zeta} \left( \frac{u^2}{v^2} \right)^{m-1}, \quad (7)$$

where the subindex of  $u_1$  has been dropped and  $A = \alpha W^{m-2}$ .

Let us introduce two new quantities defined by

$$q = \frac{v}{u}, \quad (8)$$

$$Y = \int \frac{d\zeta}{u^2} \quad (9)$$

and notice that the Wronskian type quantity built from the solutions  $u$  and  $v$  is the  $q_Y$  derivative:

$$uv_\zeta - vu_\zeta = u^2 \frac{dq}{d\zeta} = \frac{dq}{dY} = q_Y. \quad (10)$$

When we substitute all the above into (7) we obtain

$$\frac{d}{d\zeta} \left( \frac{dq}{dY} \right)^2 = -\frac{A}{m-1} Y^{m-2} \frac{d}{d\zeta} q^{2-2m}. \quad (11)$$

Now, we multiply both sides of (11) by  $u^2 = \frac{d\zeta}{dY}$  and get

$$\frac{d}{dY} \left( \frac{dq}{dY} \right)^2 = -\frac{A}{m-1} Y^{m-2} \frac{d}{dY} q^{2-2m}, \quad (12)$$

which simplifies to the following Emden-Fowler (EF) equation

$$q_{YY} = AY^{m-2} q^{1-2m} \quad (13)$$

that can be considered as equivalent to the initial Reid system of order  $m$ .

According to Polyanin [8], a particular solution is

$$q_{\text{Pol}} = \lambda_m \sqrt{Y}, \quad (14)$$

with coefficient

$$\lambda_m = (-4A)^{\frac{1}{2m}}. \quad (15)$$

Due to the particular powers of  $q$  and  $Y$ , the EF equation (13) is integrable, and the general solution can be written in parametric form. The integrability of (13) is based on the integrability of the Reid systems as we will see next.

To proceed further, we separate the well-studied case  $m = 2$  from the higher-order cases  $m \geq 3$ .

*Case 1:  $m = 2$*

In this case, (13) reduces to

$$q_{YY} = \alpha q^{-3}. \quad (16)$$

To obtain the EL invariant, let us multiply (16) by  $q_Y$  and integrate once to get

$$\frac{1}{2}(q_Y)^2 = -\frac{1}{2}\alpha q^{-2} + C, \quad (17)$$

which after identifying  $C \equiv I_2$ , where  $I_2$  is the EL invariant, yields

$$I_2(q, q_Y) = \frac{1}{2} \left[ (q_Y)^2 + \alpha q^{-2} \right]. \quad (18)$$

**Lemma 1:**  $I_2$  is a constant given by the following formula

$$I_2 = \frac{1}{2} (a^2 \alpha + b^2 W^2), \quad (19)$$

where  $a$  and  $b$  are the superposition constants of the general solution  $u$ .

Proof 1: (18) can be written as

$$I_2(\zeta) = \frac{1}{2} \left[ (uv_\zeta - vu_\zeta)^2 + \alpha \left( \frac{v}{u} \right)^{-2} \right]. \quad (20)$$

Then, using the general solution  $u$  as the linear superposition  $u = au_1 + bu_2$  and

$$v(\zeta) = \sqrt{u_1^2 + \frac{\alpha}{W^2} u_2^2} \quad (21)$$

in (20), one can get the result (19).

Thus, the EL invariant is a constant depending only on the initial conditions, the constant value of the Wronskian, and the nonlinearity parameter. To the best of our knowledge, the general formula (19) for the EL invariant was not previously mentioned in the literature. Notice that from the strict mathematical viewpoint this invariant can be zero if the superposition constants are chosen such that  $\frac{b}{a} = \pm \frac{1}{W} \sqrt{-\alpha}$ .

#### Example 1:

For illustration we use a parametric oscillator with parabolic frequency  $\omega(\zeta) = 2\zeta^2$ . This case can be found in the list of analytic cases given by Eliezer and Gray [9]. The two linearly independent solutions of the linear parametric equation in (1), with Wronskian  $W = 1$ , are

$$u_1(\zeta) = \frac{\Gamma(\frac{5}{6})}{3^{\frac{1}{6}}} \sqrt{\zeta} J_{-\frac{1}{6}} \left( \frac{2\zeta^3}{3} \right), \quad u_2(\zeta) = \frac{\Gamma(\frac{1}{6})}{2 \cdot 3^{\frac{5}{6}}} \sqrt{\zeta} J_{\frac{1}{6}} \left( \frac{2\zeta^3}{3} \right). \quad (22)$$

We choose this example because it contains nontrivial solutions that despite of their complexity will reduce to a constant when one calculates the invariant.

First, we compute the quantities  $q$  and  $q_Y$ , with  $\lambda_2 = \sqrt{2} \sqrt[4]{-\alpha}$

$$q = \frac{2\sqrt{3} \sqrt{3\alpha\zeta\Gamma^2(\frac{7}{6}) J_{\frac{1}{6}}^2(\frac{2\zeta^3}{3}) + \sqrt[3]{3}\Gamma^2(\frac{5}{6}) J_{-\frac{1}{6}}^2(\frac{2\zeta^3}{3})}}{\sqrt{\zeta} \left( 2\sqrt[3]{9}a\Gamma(\frac{5}{6}) J_{-\frac{1}{6}}(\frac{2\zeta^3}{3}) + b\Gamma(\frac{1}{6}) J_{\frac{1}{6}}(\frac{2\zeta^3}{3}) \right)} \quad (23)$$

$$q_Y = \frac{\sqrt[3]{3} \left( a\alpha\zeta {}_0F_1\left(-; \frac{7}{6}; -\frac{\zeta^6}{9}\right) - b {}_0F_1\left(-; \frac{5}{6}; -\frac{\zeta^6}{9}\right) \right)}{\sqrt{3\alpha\zeta\Gamma^2(\frac{7}{6}) J_{\frac{1}{6}}^2(\frac{2\zeta^3}{3}) + \sqrt[3]{3}\Gamma^2(\frac{5}{6}) J_{-\frac{1}{6}}^2(\frac{2\zeta^3}{3})}}, \quad (24)$$

where to keep short the numerator of the last formula we have used the definition of the Bessel functions in terms of the hypergeometric function of one parameter,  $J_n(z) = \frac{(z/2)^n}{\Gamma(1+n)} {}_0F_1(-; 1+n; -z^2/4)$ , for  $n$  not a negative integer and  $J_n(z) = (-1)^n J_{-n}(z)$  for  $n$  a negative integer [10]. Once we insert formulas (23) and (24) in (18), upon simplification gives

$$I_2 = \frac{1}{2} (a^2 \alpha + b^2) , \quad (25)$$

which corresponds to the result of **Lemma 1** for  $W = 1$ .

We turn now to the solutions of equation (16). For this, we decode the equation as the nonlinear Ermakov equation in the particular case of zero frequency  $\omega(Y) = 0$  that leads further to solutions of (1) in known forms. Since (16) is

an Ermakov equation, a particular solution can be written in terms of the two linearly independent solutions  $\tilde{q}_1 = 1$  and  $\tilde{q}_2 = Y$  of the homogeneous equation  $\tilde{q}_{YY} = 0$  using the Pinney formula [2]

$$q_{\text{Pin}}(Y) = \sqrt{\tilde{q}_1^2 + \frac{\alpha}{W^2} \tilde{q}_2^2} \equiv \sqrt{1 + \alpha Y^2} . \quad (26)$$

But knowledge of  $q(Y)$  implies getting  $v$  from

$$v_{\text{Pin}} = u_1 q_{\text{Pin}}(Y) \equiv \sqrt{u_1^2 + \alpha u_2^2} , \quad (27)$$

which is the typical Pinney formula when  $W = 1$ .

On the other hand, Polyanin's particular solution (14) corresponds to:

$$v_{\text{Pol}} = u_1 q_{\text{Pol}}(Y) = (-\alpha)^{\frac{1}{4}} \sqrt{2u_1 u_2} . \quad (28)$$

Of course, the particular solutions (27) and (28) can be obtained from the general solution of (16). Suppose we consider now the solutions  $\tilde{q}_1 = 1$  and  $\tilde{q}_2 = Y$  of Wronskian  $W = 1$  of the homogeneous equation  $\tilde{q}_{YY} = 0$ . Then, it is known that the general Pinney solution of (16) can be written as follows [11, 12]:

$$q_g(Y) = \sqrt{\alpha_1 + \alpha_2 Y^2 + 2\alpha_3 Y} , \quad (29)$$

with the  $\alpha$  constants fulfilling the condition  $\alpha_1 \alpha_2 - \alpha_3^2 = \frac{\alpha}{W^2}$ . One can easily see that  $q_{\text{Pin}}(Y)$  and  $q_{\text{Pol}}(Y)$  are just particular cases of  $q_g(Y)$ .

*Case 2:  $m > 2$*

For the general case, to find the invariant we will use two methods described below.

*The Kamke route*

First, we use the substitutions in Kamke's book [13],  $q(Y) = y(x)$  and  $x = \frac{1}{Y}$ , in equation (13) that lead to the Emden-Fowler equation in the normal form:

$$y'' + \frac{2}{x} y' = A x^{-m-2} y^{1-2m} . \quad (30)$$

Furthermore, the substitution  $y = \frac{V(x)}{\sqrt{x}}$  singles out the nonlinear term

$$x^2 V'' + x V' - \frac{1}{4} V = A V^{1-2m} . \quad (31)$$

To get rid of the damping term, one can use the Euler exponential change of independent variable  $x = e^\eta$  that leads to

$$V_{\eta\eta} - \frac{1}{4} V = A V^{1-2m} , \quad (32)$$

which is a Reid equation of constant frequency  $\omega = \frac{i}{2}$  and therefore of known solution [1]

$$V(\eta) = \left( e^{+\frac{m\eta}{2}} + \frac{A}{m-1} e^{-\frac{m\eta}{2}} \right)^{\frac{1}{m}} , \quad (33)$$

where the exponential functions are the linear independent solutions of Wronskian  $W = -1$  of the hyperbolic oscillator equation

$$U_{\eta\eta} - \frac{1}{4} U = 0 . \quad (34)$$

Multiplying (32) by  $V_\eta$  and integrating, one immediately gets

$$V_\eta^2 = \frac{1}{4} V^2 - \frac{A}{m-1} V^{2-2m} + I_m \equiv R(V) , \quad (35)$$

which provides the general expression for the Ermakov-Lewis invariant for  $m > 2$ :

$$I_m(V, V_\eta) = V_\eta^2 + \frac{A}{m-1} V^{2-2m} - \frac{1}{4} V^2. \quad (36)$$

Equation (36) can be also written as a function of corresponding Reid's  $q$ 's and  $q_Y$ 's as follows:

$$I_m(q, q_Y) = Y(q_Y)^2 - q_Y q + \frac{A}{m-1} \left( \frac{Y}{q^2} \right)^{m-1}. \quad (37)$$

In terms of  $u, v$  the above becomes

$$I_m(\zeta) = (uv_\zeta - vu_\zeta)^2 \int \frac{d\zeta}{u^2} - \frac{v}{u} (uv_\zeta - vu_\zeta) + \frac{\alpha W^{m-2}}{m-1} \left( \frac{u^2}{v^2} \int \frac{d\zeta}{u^2} \right)^{m-1}, \quad (38)$$

which is the higher order equivalent of (20).

If in (37) we now substitute the Polyanin particular solution we find the following constant value for the Reid invariant for all  $m \geq 3$ :

$$I_m = -\frac{\lambda_m^2}{4} \left[ 1 + \frac{1}{m-1} \right] \equiv -\frac{(-4A)^{\frac{1}{m}}}{4} \frac{m}{m-1}. \quad (39)$$

We now address the issue of finding the Reid solution (2) from the solution of the EF equation (13) for  $m > 2$ . Formula (35) is separable as follows:

$$\int \frac{\pm dV}{\sqrt{R(V)}} = \int d\eta = \eta - \eta_0 = \ln|x| - \ln|x_0| = \ln \left| \frac{x}{x_0} \right|, \quad (40)$$

which allows us to introduce the exponential parametric form of the Emden-Fowler solutions:

$$x(V) = |x_0| e^{\pm \int^V \Theta(V') dV'} \quad (41)$$

$$y(V) = \frac{V}{\sqrt{|x_0|}} e^{\mp \frac{1}{2} \int^V \Theta(V') dV'} \quad (42)$$

where  $\Theta(V) = R^{-\frac{1}{2}}(V)$ . In our case the solutions are:

$$Y_m(V) = \frac{1}{|x_0|} e^{\mp \int^V \left( \frac{1}{4} V'^2 - \frac{A}{m-1} V'^{2-2m} + I_m \right)^{-\frac{1}{2}} dV'} \quad (43)$$

$$q_m(V) = \frac{V}{\sqrt{|x_0|}} e^{\mp \frac{1}{2} \int^V \left( \frac{1}{4} V'^2 - \frac{A}{m-1} V'^{2-2m} + I_m \right)^{-\frac{1}{2}} dV'}, \quad (44)$$

where  $x_0$  is a constant of integration. For example, in the  $m = 3$  case one gets  $I_3 = \frac{3}{4} \left( \frac{A}{2} \right)^{\frac{1}{3}}$ , and then the parametric solutions to the EF equation are as follows:

$$Y_3(V) = \frac{1}{|x_0|} e^{\mp \int^V \left( \frac{1}{4} V'^2 - \frac{A}{2} V'^{-4} + \frac{3}{4} \left( \frac{A}{2} \right)^{\frac{1}{3}} \right)^{-\frac{1}{2}} dV'} \quad (45)$$

$$q_3(V) = \frac{V}{\sqrt{|x_0|}} e^{\mp \frac{1}{2} \int^V \left( \frac{1}{4} V'^2 - \frac{A}{2} V'^{-4} + \frac{3}{4} \left( \frac{A}{2} \right)^{\frac{1}{3}} \right)^{-\frac{1}{2}} dV'}. \quad (46)$$

From (43) and (44) it follows that  $q$  and  $Y$  for higher order Reid systems are connected by

$$q = V \sqrt{Y} \quad (47)$$

and therefore the parametric solutions differ from the particular Polyanin solution through the function  $V$  that replaces the constant factor  $\lambda_m$ . Moreover, using  $e^\eta = x = \frac{1}{Y}$  in (33) one gets

$$V(Y) = \left( Y^{-\frac{m}{2}} + \frac{A}{m-1} Y^{\frac{m}{2}} \right)^{\frac{1}{m}}. \quad (48)$$

Then (47) gives

$$v = Vu_1\sqrt{Y} = V\sqrt{\frac{u_1u_2}{W}} = \left(Y^{-\frac{m}{2}} + \frac{A}{m-1}Y^{\frac{m}{2}}\right)^{\frac{1}{m}}\sqrt{\frac{u_1u_2}{W}}. \quad (49)$$

Since  $Y = \frac{u_2}{Wu_1}$  we finally get

$$v = \left(u_1^m + \frac{\alpha}{(m-1)W^2}u_2^m\right)^{\frac{1}{m}}, \quad (50)$$

which is Reid's formula for solution  $v$ .

#### *The Abel equation route*

One can also solve the full Emden-Fowler equation by the reduction of order method using both dependent-independent variable substitutions

$$z = \left(\frac{Y}{q^2}\right)^m, \quad (51)$$

$$w = Y\frac{q_Y}{q} \quad (52)$$

that lead to a first order Abel equation

$$\left(w - \frac{1}{2}\right)w_z = \frac{w^2}{2mz} - \frac{w}{2mz} - \frac{A}{2m}. \quad (53)$$

Now, if we let  $w - \frac{1}{2} = \frac{1}{w}$  we obtain the Bernoulli equation

$$w_z = -\frac{1}{2mz}w + \frac{1+4Az}{8mz}w^3. \quad (54)$$

This can be linearized by  $\varphi = w^{-2}$  to give

$$\varphi_z - \frac{1}{mz}\varphi = -\frac{A}{m} - \frac{1}{4mz}. \quad (55)$$

The solution to (55) is

$$\varphi(z) = \frac{A}{1-m}z + I_m z^{\frac{1}{m}} + \frac{1}{4}, \quad (56)$$

where  $I_m$ ,  $m > 2$  is an integration constant which is equivalent with the Reid Invariant.

Now we use back all the substitutions and solve for  $I_m$  to obtain

$$I_m = Y(q_Y)^2 - q_Y q + \frac{\alpha W^{m-2}}{m-1} \left(\frac{Y}{q^2}\right)^{m-1}, \quad (57)$$

which is the same as equation (37) obtained previously.

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